Pergamon

Int. J. Solids Structures Vol. 34, No. 27, pp. 3583–3599, 1997

0020-7683/97 $17.00 + .00

PII: S0020-7683(96)00192-8

MODELLING OF WEAKLY BONDED LAMINATED COMPOSITE PLATES AT LARGE DEFLECTIONS

ZHEN-QIANG CHENG†
Department of Modern Mechanics, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China

and

W. P. HOWSON and F. W. WILLIAMS
Division of Structural Engineering, Cardiff School of Engineering, University of Wales Cardiff, Cardiff CF2 1XH, U.K.

(*Received* 1 *August* 1995; *in revised form* 13 *August* 1996)

Abstract—Based on a general representation of displacement variation through the thickness of laminated plates, a Karman type nonlinear theory of laminated composite plates with weakened interfacial bonding is developed. Each weakly bonded interface is modelled by a spring-layer model which has recently been used efficiently in the field of micromechanics of composites. This spring-layer model allows for a discontinuous distribution of displacements, but requires the tractions to be continuous across each interface of adjacent layers. The set of governing equations has variable coefficients in the most general form of bonding and includes conventional third-order zigzag nonlinear theory of Karman type for laminated composite plates as a special case when extreme values of interface parameters are used. Some simple numerical examples allowing for a closed-form solution are presented to give an understanding of how a small amount of interfacial weakness affects the overall and local behaviour of laminated composite plates. These include the important practical problem of reduced interface stresses due to weakened interfacial bonding, which can be predicted by the theory presented herein. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

It is well known that gross theories of laminated composite plates and shells fail to predict local elastic responses at the ply level sufficiently accurately. This is because continuity of tractions is not imposed on interfaces of adjacent laminae, despite the fact that transverse shear stress is one of the most important causes of delamination in laminated structures. To overcome this problem, various zigzag theories, alternatively termed simplified discrete-layer theories (Noor and Burton, 1989) or refined single-layer theories (Reddy and Robbins Jr, 1994) have recently been proposed, see Di Sciuva (1986, 1987, 1992), Di Sciuva and Icardi (1993), Savithri and Varadan (1990, 1993), Librescu and Schmidt (1991), Gaudenzi (1992), Cho and Parmerter (1992, 1993, 1994), Xavier *et al.* (1993), He (1993, 1994, 1995) and Schmidt and Librescu (1994). The displacement field assumed is such that the displacements and tractions are continuous at layer interfaces. This continuity can be used to reduce the total number of unknown parameters in the theories. Such approaches formulate a multilayered plate model of the discrete-layer category for which the total number of generalized displacements does not increase with the number of layers. This number is usually five, as in most smeared theories such as first-order equivalent single-layer theory or Reddy's (1984) third-order theory.

In contrast to their metallic counterparts, the anisotropic constitution of laminated composite structures often results in unique phenomena that can occur at vastly different geometric scales, i.e., at the global level, the ply level or the reinforcement-matrix level. The equivalent single-layer theories are generally capable of describing the global response sufficiently accurately, whereas at the ply level discrete-layer and zigzag theories are needed

† Academic Visitor, Division of Structural Engineering, Cardiff School of Engineering, University of Wales Cardiff, Cardiff CF2 1XH, U.K.

to determine the three-dimensional stress field. At the reinforcement-matrix level, micromechanics of composites has emerged as a critical research area investigating the properties of random or deterministic heterogeneous particulate and fibrous composite materials. The present paper proposes a geometrically nonlinear theory which is completely general when applied at ply level and only requires that the elasticity constants of each layer have been determined either by experiment or from micromechanics techniques. Particular attention is paid to multilayered anisotropic plates with imperfect layer interfaces.

It has been widely observed that the behaviour of composite materials is significantly influenced by the properties of the interfaces between the constituents. A perfect interface, which implies continuity of displacements and tractions across the interface, is assumed in the majority of work on composite materials, with the result that interface properties and structures are eliminated. However, there are many applications where the assumption of a perfect interface is inadequate. Examples for laminated composites could be the presence of a thin layer between adjacent ply layers or a coating on the surface of the reinforcing constituent. Such an interfacial layer is generally referred to as an interphase. Such interphases may be introduced to inhibit chemical interaction between the constituents or to improve the properties of the composite. In the limit of vanishing interphase-thickness, displacement jumps occur when crossing the interphase from one side to the other, while the tractions must remain continuous from simple considerations of equilibrium. The simplest way of representing this is to assume that the jumps in normal and tangential displacements are proportional to the corresponding tractions, giving a spring-layer model. Such a model has recently been applied in micromechanics-based research on imperfect interfaces of composites at the reinforcement matrix level, e.g., see Benveniste (1985), Aboudi (1987), Achenbach and Zhu (1989), Jasiuk and Tong (1989), Benveniste and Dvorak (1990), Hashin (1990, 1991a,b, 1993) and Qu (1993a,b). However, this paper concerns itself with one area which has received very little attention and yet has significant ramifications for practical structures, namely the effects of weak bonding at the ply level of laminated composites. See also Mao and Han (1992) and Schmidt and Librescu (1995).

The theory presented incorporates the most important interfacial properties into a geometrically nonlinear theory of Karman type for multilayered anisotropic plates. Each interface between adjacent layers utilises the spring-layer model employed in micromechanics. An important feature of this paper is the use of such a model in macrostructural analysis. As will be shown, the use of this model in the two-dimensional theory of multilayered plates and shells avoids the physically impossible phenomenon of interpenetration at the interfaces. However, this model might lead to such interpenetration within an elastic, three-dimensional approach, as discussed briefly by Achenbach and Zhu (1989), although more accurate models may be found to overcome this problem. An approximate displacement model is given which includes displacement jumps across each interface and thus enables interfacial imperfection to be incorporated. As it satisfies the compatibility conditions for transverse shear stresses, both at layer interfaces and on the two bounding surfaces of the plate, there is no need for the use of shear correction factors. Furthermore, the number of unknowns is eventually shown to be five, irrespective of the number of layers. The set of governing equations has variable coefficients when considering non-uniform bonding strength at each interface and constant coefficients when the bonding is uniform. In the limit of vanishing interface parameters, this theory in linear dynamic or nonlinear static form reduces to exactly the flat plate limit of zigzag theory for multilayered anisotropic shells given by He (1994, 1995). Some simple numerical examples are presented to illustrate the effects of a small amount of interface weakness on the overall and local behaviour of multilayered plates.

2. APPROXIMATE DISPLACEMENT FIELD

Figure 1 shows a multilayered plate consisting of k homogeneous anisotropic layers of uniform thickness. The undeformed lower surface of the plate is chosen as the reference surface defined by $x_3 = 0$ and the x_3-axis is normal to it, where $\{x_i\}$ $(i = 1, 2, 3)$ is a Cartesian coordinate system. Let ${}^{(m)}\Omega$ $(m = 0, \ldots, k)$ denote, respectively, the lower surface $(m = 0)$,

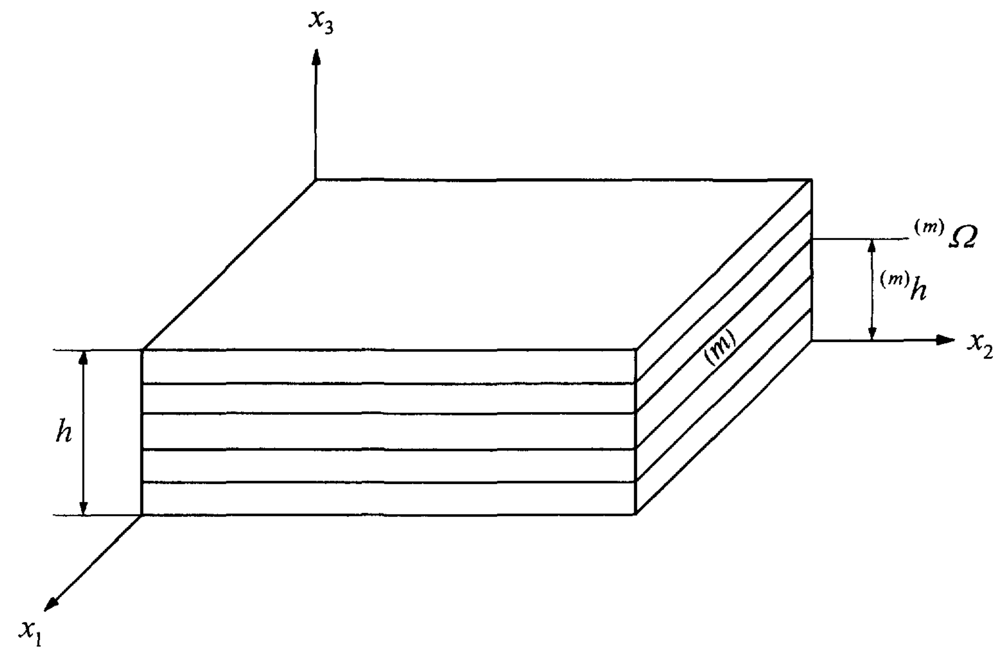

Fig. 1. Geometry of a laminated plate.

the interface between the mth and $(m+1)$th layers $(m = 1, \ldots, k-1)$ and the upper surface $(m = k)$ of the plate. The range of the mth layer in the x_3-direction is $[^{(m-1)}h, {}^{(m)}h]$, where $^{(m)}h$ $(m = 0, \ldots, k)$ is the distance between $^{(0)}\Omega$ and $^{(m)}\Omega$. Clearly, $^{(0)}h = 0$ and $^{(k)}h = h$, where h is the total thickness of the plate.

Throughout the following derivations, a comma followed by a subscript denotes a derivative with respect to the corresponding spatial coordinate, and a dot over a quantity refers to a derivative with respect to time, t. The Einsteinian summation convention applies to repeated subscripts of tensor components, with English subscripts ranging from 1 to 3 while Greek subscripts are either 1 or 2. The spatial derivative of the Heaviside step function $H(x_3 - {}^{(m)}h)$ with respect to x_3 is stipulated as the right-hand one, thus $H_{,3}(x_3 - {}^{(m)}h) = 0$.

Following He's (1994, 1995) general representation of displacement variation, the displacement $v_j(x_i; t)$ of any point in the plate can be expressed as

$$v_j(x_i; t) = \sum_{m=0}^{k-1} \sum_{n=0}^{\infty} {}^{(m)}u_j^{(n)}(x_\alpha; t)(x_3 - {}^{(m)}h)^n H(x_3 - {}^{(m)}h), \tag{1}$$

where the following term has been retained in the present theory, but excluded for perfect interfaces by He (1994, 1995). Denoting $^{(0)}v_j(x_i; t) \equiv 0$, yields

$$^{(m)}u_j^{(0)}(x_\alpha; t) = {}^{(m+1)}v_j(x_\alpha, {}^{(m)}h; t) - {}^{(m)}v_j(x_\alpha, {}^{(m)}h; t). \tag{2}$$

This term implies that the displacements at interfaces are allowed to be discontinuous, so as to provide a possible incorporation of imperfect interfaces of multilayered plates, e.g., weakened bonding or even delamination. The case of perfect bonding corresponds to this term being zero.

To incorporate the properties and structures of interfaces in the evaluation of composite behaviour, interfaces must be treated as regions of distinct atomic structure and, possibly, distinct composition. They should have different properties from the bulk properties on either side of the interface. In the context of continuum mechanics, one simple approach is to introduce a thin layer of interphase material which replaces the interface. The limiting case of vanishing interphase-thickness then gives an interface which is a mathematical surface across which material properties change discontinuously, with the interfacial tractions being continuous while the displacements are discontinuous. Although nonlinear relationships may be proposed between the interfacial tractions and displacement jumps, a linear spring-layer model is explored in this paper to characterize the imperfect bonding. Thus

$$\sigma_{\alpha 3}(x_\alpha, {}^{(m)}h^+;t) = \sigma_{\alpha 3}(x_\alpha, {}^{(m)}h^-;t), \quad \sigma_{33}(x_\alpha, {}^{(m)}h^+;t) = \sigma_{33}(x_\alpha, {}^{(m)}h^-;t), \quad (m = 1,\ldots,k-1), \quad \text{(3a,b)}$$

$${}^{(m)}\Delta v_\alpha = {}^{(m)}R_{\alpha\beta}(x_\rho)\sigma_{\beta 3}(x_\alpha, {}^{(m)}h;t), \quad {}^{(m)}\Delta v_3 = {}^{(m)}R_{33}(x_\rho)\sigma_{33}(x_\alpha, {}^{(m)}h;t), \quad (m = 1,\ldots,k-1), \quad \text{(4a,b)}$$

where ${}^{(m)}R_{\alpha\beta}$ and ${}^{(m)}R_{33}$ in eqn (4) represent the compliance coefficients of the spring-layer interface ${}^{(m)}\Omega$. It is clear from eqn (4) that ${}^{(m)}R_{\alpha\beta} = 0$ and ${}^{(m)}R_{33} = 0$ correspond to a perfect interface, while ${}^{(m)}R_{\alpha\beta} \to \infty$ and ${}^{(m)}R_{33} \to \infty$ represent complete debonding, i.e., $\sigma_{i3} = 0$ on ${}^{(m)}\Omega$. From this point of view, a slightly weakened interface may be modelled by small values of ${}^{(m)}R_{\alpha\beta}$ and ${}^{(m)}R_{33}$. Such an imperfect interface may be due to the presence of an interphase but could also be due to interface bond deterioration caused by, e.g., fatigue damage or environmental and chemical effects.

When ${}^{(m)}R_{33} = 0$, this constitutive characterization of the interface allows relative sliding between the two surfaces, but no separation. Furthermore, the free-sliding case can be achieved by setting ${}^{(m)}R_{\alpha\beta} \to \infty$ with ${}^{(m)}R_{33} = 0$. It should be noted that when ${}^{(m)}R_{33} \neq 0$ this mathematical model includes solutions which are physically impossible because one constituent would have to penetrate another, as noticed by Achenbach and Zhu (1989) and Qu (1993a,b). This violates the compatibility requirements and therefore the model is apparently unreasonable for such a case. Fortunately, the normal stress σ_{33} for the plate problem under consideration is assumed to be negligibly small compared with other stress components, so that it is ignored in the present theory as in most other theories for plates and shells. This automatically leads to an identity eqn (3b) and a vanishing displacement jump ${}^{(m)}\Delta v_3$, see eqn (4b), regardless of the value of the interface parameter ${}^{(m)}R_{33}$. Therefore it seems to be reasonable to adopt this spring-layer model in the theory of plates and shells with imperfect bonding in shear.

Under the assumptions that the normal stress σ_{33} for the plate problem is negligibly small compared with other stress components and that the material symmetry is that of reflectional symmetry in planes parallel to the reference plane, an approximate displacement field is given as (see eqns (A10) and (A6) in Appendix A)

$$v_\alpha(x_i;t) = u_\alpha - x_3 u_{3,\alpha} + f_{\alpha\lambda}\varphi_\lambda + \sum_{m=1}^{k-1} {}^{(m)}\Delta v_\alpha H(x_3 - {}^{(m)}h), \quad v_3(x_i;t) = u_3, \quad \text{(5a,b)}$$

where u_j, φ_λ and ${}^{(m)}\Delta v_\alpha$ are independent of x_3. The development details of the displacement field (5) and the expression for $f_{\alpha\lambda}$ are given in Appendix A. The only difference of the assumed displacement distribution along the thickness from that for perfect bonding is the incorporation of the displacement jump ${}^{(m)}\Delta v_\alpha$ across each layer interface ${}^{(m)}\Omega$, which is determined as

$${}^{(m)}\Delta v_\alpha = {}^{(m)}R_{\alpha\beta}(x_\rho)\,{}^{(m+1)}E_{\beta 3\omega 3} f_{\omega\lambda,3}({}^{(m)}h^+)\varphi_\lambda, \quad \text{(6)}$$

from eqns (5), (A3a,c) and (4a). Substituting into eqn (5a) gives the approximate displacement expression

$$v_\alpha = u_\alpha - x_3 u_{3,\alpha} + h_{\alpha\lambda}\varphi_\lambda, \quad \text{(7)}$$

in which

$$h_{\alpha\lambda} \equiv h_{\alpha\lambda}(x_i) = f_{\alpha\lambda}(x_3) + \sum_{m=1}^{k-1} {}^{(m)}R_{\alpha\beta}(x_\rho)^{(m+1)}E_{\beta 3\omega 3}f_{\omega\lambda,3}({}^{(m)}h^+)H(x_3 - {}^{(m)}h). \tag{8}$$

The fact that the interface parameter ${}^{(m)}R_{\alpha\beta}$ depends upon x_ρ implies that the bonding strength at the interface ${}^{(m)}\Omega$ $(m = 1, \ldots, k-1)$ may be non-uniform, i.e., general cases of a small amount of interface weakness are included in the present theory.

By using the displacement expressions of eqns (7) and (5b), the associated strain and stress components can be obtained from eqns (A3), but their explicit forms are not given herein.

3. BOUNDARY VALUE PROBLEM

It is assumed that the mass density ρ of the plate is independent of time t and that an arbitrarily distributed normal load $q(x_\alpha; t)$ is applied to the surface Ω (${}^{(0)}\Omega$ or ${}^{(k)}\Omega$). From Hamilton's principle

$$\int_0^{t_0} \left(\int_V \sigma_{ij}\delta e_{ij}\,\mathrm{d}V - \int_V \dot{v}_i\delta\dot{v}_i\rho\,\mathrm{d}V - \int_\Omega q\delta v_3\,\mathrm{d}\Omega \right)\mathrm{d}t = 0, \tag{9}$$

the nonlinear dynamic fundamental equations are derived as

$$\begin{aligned} N_{\alpha\beta,\beta} - I\ddot{u}_\alpha + J\ddot{u}_{3,\alpha} - I_{\alpha\beta}\ddot{\varphi}_\beta &= 0, \\ M_{\alpha\beta,\alpha\beta} + q + (N_{\alpha\beta}u_{3,\alpha})_{,\beta} - I\ddot{u}_3 - J\ddot{u}_{\alpha,\alpha} + K\ddot{u}_{3,\alpha\alpha} - (J_{\alpha\beta}\ddot{\varphi}_\beta)_{,\alpha} &= 0, \\ P_{\lambda\beta,\beta} - R_\lambda - I_{\alpha\lambda}\ddot{u}_\alpha + J_{\alpha\lambda}\ddot{u}_{3,\alpha} - K_{\beta\lambda}\ddot{\varphi}_\beta &= 0, \end{aligned} \tag{10}$$

associated with either one of each of the following pairs of boundary conditions

$$\begin{aligned} n_\beta N_{\alpha\beta} &= 0, \quad \text{or} \quad \delta u_\alpha = 0, \\ n_\beta(M_{\alpha\beta,\alpha} + N_{\alpha\beta}u_{3,\alpha} - J\ddot{u}_\beta + K\ddot{u}_{3,\beta} - J_{\beta\alpha}\ddot{\varphi}_\alpha) &= 0, \quad \text{or} \quad \delta u_3 = 0, \\ n_\beta P_{\alpha\beta} &= 0, \quad \text{or} \quad \delta\varphi_\alpha = 0, \\ n_\beta M_{\alpha\beta} &= 0, \quad \text{or} \quad \delta u_{3,\alpha} = 0, \end{aligned} \tag{11}$$

where

$$[N_{\alpha\beta}, M_{\alpha\beta}, P_{\lambda\beta}] = \int_0^h \sigma_{\alpha\beta}[1, x_3, h_{\alpha\lambda}]\,\mathrm{d}x_3, \tag{12}$$

$$R_\lambda = \int_0^h \sigma_{\alpha i}h_{\alpha\lambda,i}\,\mathrm{d}x_3, \tag{13}$$

$$[I, J, K] = \int_0^h \rho[1, x_3, x_3^2]\,\mathrm{d}x_3, \tag{14}$$

$$[I_{\alpha\beta}, J_{\alpha\beta}, K_{\lambda\beta}] = \int_0^h \rho h_{\alpha\beta}[1, x_3, h_{\alpha\lambda}]\,\mathrm{d}x_3. \tag{15}$$

Furthermore, eqns (12) and (13) can be rewritten as, by using eqns (7), (5b) and (A3),

$$\begin{bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\lambda\beta} \\ R_{\lambda} \end{bmatrix} = \begin{bmatrix} C^{(1)}_{\alpha\beta\omega\rho} & -C^{(2)}_{\alpha\beta\omega\rho} & C^{(3)}_{\alpha\beta\nu\rho} & C^{(3)}_{\alpha\beta\nu\rho,\rho} \\ C^{(2)}_{\alpha\beta\omega\rho} & -C^{(4)}_{\alpha\beta\omega\rho} & C^{(5)}_{\alpha\beta\nu\rho} & C^{(5)}_{\alpha\beta\nu\rho,\rho} \\ C^{(3)}_{\omega\rho\lambda\beta} & -C^{(5)}_{\omega\rho\lambda\beta} & C^{(6)}_{\lambda\beta\nu\rho} & C^{(7)}_{\lambda\beta\nu} \\ C^{(3)}_{\omega\rho\lambda\beta,\beta} & -C^{(5)}_{\omega\rho\lambda\beta,\beta} & C^{(7)}_{\nu\rho\lambda} & C^{(8)}_{\lambda\nu} \end{bmatrix} \begin{bmatrix} u_{\omega,\rho}+\frac{1}{2}u_{3,\omega}u_{3,\rho} \\ u_{3,\omega\rho} \\ \varphi_{\nu,\rho} \\ \varphi_{\nu} \end{bmatrix}, \quad (16)$$

where

$$[C^{(1)}_{\alpha\beta\omega\rho}, C^{(2)}_{\alpha\beta\omega\rho}, C^{(3)}_{\alpha\beta\nu\rho}, C^{(4)}_{\alpha\beta\omega\rho}, C^{(5)}_{\alpha\beta\nu\rho}, C^{(6)}_{\lambda\beta\nu\rho}, C^{(7)}_{\lambda\beta\nu}] = \int_0^h H_{\alpha\beta\omega\rho}[1, x_3, h_{\omega\nu}, x_3^2, x_3 h_{\omega\nu}, h_{\alpha\lambda}h_{\omega\nu}, h_{\alpha\lambda}h_{\omega\nu,\rho}]\,\mathrm{d}x_3, \quad (17)$$

$$C^{(8)}_{\lambda\nu} = \int_0^h (H_{\alpha\beta\omega\rho}h_{\alpha\lambda,\beta}h_{\omega\nu,\rho} + E_{\alpha 3\omega 3}h_{\alpha\lambda,3}h_{\omega\nu,3})\,\mathrm{d}x_3. \quad (18)$$

Finally, substitution of eqns (16) into eqns (10) yields

$$C^{(1)}_{\alpha\beta\omega\rho}(u_{\omega,\rho}+\tfrac{1}{2}u_{3,\omega}u_{3,\rho})_{,\beta} - C^{(2)}_{\alpha\beta\omega\rho}u_{3,\omega\rho\beta} + (C^{(3)}_{\alpha\beta\nu\rho}\varphi_{\nu})_{,\rho\beta} - I\ddot{u}_{\alpha} + J\ddot{u}_{3,\alpha} - I_{\alpha\beta}\ddot{\varphi}_{\beta} = 0,$$

$$\begin{aligned} &C^{(2)}_{\alpha\beta\omega\rho}(u_{\omega,\rho}+\tfrac{1}{2}u_{3,\omega}u_{3,\rho})_{,\alpha\beta} - C^{(4)}_{\alpha\beta\omega\rho}u_{3,\omega\rho\alpha\beta} + (C^{(5)}_{\alpha\beta\nu\rho}\varphi_{\nu})_{,\rho\alpha\beta} + q \\ &\quad + \{[C^{(1)}_{\alpha\beta\omega\rho}(u_{\omega,\rho}+\tfrac{1}{2}u_{3,\omega}u_{3,\rho}) - C^{(2)}_{\alpha\beta\omega\rho}u_{3,\omega\rho} + (C^{(3)}_{\alpha\beta\omega\rho}\varphi_{\omega})_{,\rho}]u_{3,\alpha}\}_{,\beta} \\ &\quad - I\ddot{u}_3 - J\ddot{u}_{\alpha,\alpha} + K\ddot{u}_{3,\alpha\alpha} - (J_{\alpha\beta}\ddot{\varphi}_{\beta})_{,\alpha} = 0, \end{aligned}$$

$$\begin{aligned} &C^{(3)}_{\omega\rho\lambda\beta}(u_{\omega,\rho\beta}+\tfrac{1}{2}u_{3,\omega}u_{3,\rho})_{,\beta} - C^{(5)}_{\omega\rho\lambda\beta}u_{3,\omega\rho\beta} + C^{(6)}_{\lambda\beta\nu\rho}\varphi_{\nu,\rho\beta} + (C^{(6)}_{\lambda\beta\nu\rho,\beta} + C^{(7)}_{\lambda\rho\nu} - C^{(7)}_{\nu\rho\lambda})\varphi_{\nu,\rho} \\ &\quad + (C^{(7)}_{\lambda\beta\nu,\beta} - C^{(8)}_{\lambda\nu})\varphi_{\nu} - I_{\alpha\lambda}\ddot{u}_{\alpha} + J_{\alpha\lambda}\ddot{u}_{3,\alpha} - K_{\beta\lambda}\ddot{\varphi}_{\beta} = 0. \quad (19) \end{aligned}$$

These equations need to be solved with the boundary conditions of eqns (11) to obtain the unknowns u_{α}, u_3 and φ_{α} for any set of plate parameters and the load parameter q. Obviously, eqns (19) have variable coefficients simply due to the non-uniform value of interface parameters ${}^{(m)}R_{\alpha\beta}$ at the interfaces ${}^{(m)}\Omega$ ($m = 1, \ldots, k-1$), while for problems with uniform bonding strength at each interface, eqns (19) will have constant coefficients. By setting ${}^{(m)}R_{\alpha\beta} = 0$ ($m = 1, \ldots, k = 1$), the corresponding governing equations and boundary conditions become simply those for perfect bonding. In the linear dynamic or nonlinear static case, they are exactly the same as those given by collapsing the shell theory of He (1994, 1995) to the corresponding plate case, and are also very similar to those proposed by Di Sciuva (1992).

4. ILLUSTRATIVE EXAMPLE

The theory presented offers the opportunity to solve a wide range of complicated problems. However, complete solutions to such problems require the determination of interface parameters either through theoretical evaluation of interfacial properties and microstructures or experimental measurements. Since the evaluation of such parameters is beyond the scope of this paper, an insight into the influence of interfacial weakness on the global and local behaviour of multilayered anisotropic plates will be given by restricting attention to the effects of slightly weakened interfaces on their linear bending and vibration behaviour. A rectangular orthotropic three-ply symmetric laminated plate of length a and width b will be used as the example for analyzing such interfacial weakness. The plate is simply supported at edges $x_1 = 0$, a and $x_2 = 0$, b. Identically uniform bonding of the interfaces is assumed.

Under the action of $q = q_0 \sin(m_1\pi x_1/a)\sin(m_2\pi x_2/b)\exp(i\omega t)$, a closed-form solution of this problem has the following form

$$[u_1, \varphi_1] = [U_1, \Phi_1]\cos\frac{m_1\pi x_1}{a}\sin\frac{m_2\pi x_2}{b}e^{i\omega t},$$

$$[u_2, \varphi_2] = [U_2, \Phi_2]\sin\frac{m_1\pi x_1}{a}\cos\frac{m_2\pi x_2}{b}e^{i\omega t},$$

$$u_3 = U_3\sin\frac{m_1\pi x_1}{a}\sin\frac{m_2\pi x_2}{b}e^{i\omega t}. \quad (20)$$

From these expressions, exact solutions can easily be given for static bending by taking $\omega = 0$ and $m_1 = m_2 = 1$, and for flexural vibration by taking $q_0 = 0$. For brevity, an overview of the procedure for obtaining closed form solutions is given in Appendix B.

Two kinds of material were chosen for the numerical computations.

(a) Material 1: a (0°/90°/0°) laminated plate with identical layer thickness and stiffness properties

$$E_L/E_T = 25, \quad G_{LT}/E_T = 0.5, \quad G_{TT}/E_T = 0.2, \quad \nu_{LT} = \nu_{TT} = 0.25, \quad (21)$$

where E is the tensile modulus, G is the shear modulus, ν is Poisson's ratio and the subscripts L and T refer to the directions parallel and normal to the fibres, respectively.

(b) Material 2: an orthotropic three-layered plate with ${}^{(1)}h/h = 0.1$ and ${}^{(2)}h/h = 0.9$, and identical relative values of the elastic moduli for each layer as

$$E_{2222}/E_{1111} = 0.543103, \quad E_{3333}/E_{1111} = 0.530172,$$

$$E_{1122}/E_{1111} = 0.23319, \quad E_{1133}/E_{1111} = 0.010776, \quad E_{2233}/E_{1111} = 0.098276,$$

$$E_{1212}/E_{1111} = 0.262931, \quad E_{1313}/E_{1111} = 0.159914, \quad E_{2323}/E_{1111} = 0.26681. \quad (22)$$

The interface parameters are taken as ${}^{(m)}R_{\alpha\beta} = \delta_{\alpha\beta}\bar{R}h/E$ $(m = 1, 2)$, with $E = E_T$ for Material 1 and $E = {}^{(2)}E_{1111}$ for Material 2, where $\bar{R}$ is a dimensionless quantity. All of the numerical results were calculated for Material 1 except Table 2 which is based on Material 2. Table 1 shows the dimensionless central deflection and stresses for various values of $\bar{R}$, together with comparative exact results given by Pagano (1970) for perfect interfaces calculated from three-dimensional elasticity. As is well known in the literature, e.g., see Di Sciuva (1986, 1992), Cho and Parmerter (1992, 1993) and He (1994, 1995), most theories for perfectly bonded plates and shells, which make use of an *a priori* assumption of through-thickness displacement distribution, fail to predict sufficiently accurately the transverse shear stresses for moderately thick and very thick plates directly from the constitutive equations, even though interface continuity conditions of tractions and displacements have been imposed. Instead, they are evaluated accurately from the equilibrium equations. In similar fashion, the trend of curves showing variations of the interlaminar shear stresses with $\bar{R}$ calculated directly from constitutive relations seems to be physically unreasonable, see the work by Cheng *et al.* (1996). Therefore the transverse shear stresses in Table 1, as well as in Fig. 9, were calculated from the equilibrium equation $\sigma_{\alpha j,j} = 0$. Some comments on the use of the "*a posteriori*" calculation of such components by means of three-dimensional equilibrium and constitutive relations were given by Noor and Burton (1989). Table 2 gives frequency values of the plate fabricated from Material 2 when vibrating in its fundamental flexural mode. The exact three-dimensional elasticity solution obtained by Srinivas and Rao (1970) is also given for comparison. When the theory is used to consider the special case of perfect interfaces, the present results for $\bar{R} = 0$ in Tables 1 and 2 are exactly the

Table 1. Central deflection and stresses of a rectangular three-ply plate ($a/b = 1/3$, Material 1) under sinusoidal loading

	a/h	$\bar{R} = 0$ (Exact)	$\bar{R} = 0$	$\bar{R} = 0.2$	$\bar{R} = 0.4$	$\bar{R} = 0.6$
$\dfrac{100E_T h^3 v_3\left(\frac{a}{2},\frac{b}{2},\frac{h}{2}\right)}{q_0 a^4}$	4	2.820	2.75668	3.34194	3.88249	4.36220
	10	0.919	0.91966	1.06740	1.23141	1.40863
	20	0.610	0.60983	0.64949	0.69499	0.74608
	100	0.508	0.50767	0.50930	0.51118	0.51332
$\dfrac{h^2\sigma_{11}\left(\frac{a}{2},\frac{b}{2},h\right)}{q_0 a^2}$	4	1.14	1.20469	1.39003	1.56652	1.72744
	10	0.726	0.73047	0.77525	0.82590	0.88143
	20	0.650	0.65067	0.66262	0.67658	0.69244
	100	0.624	0.62437	0.62486	0.62544	0.62610
$\dfrac{h^2\sigma_{22}\left(\frac{a}{2},\frac{b}{2},\left(\frac{2h}{3}\right)^-\right)}{q_0 a^2}$	4	0.109	0.10811	0.13184	0.15391	0.17364
	10	0.0418	0.04196	0.04860	0.05604	0.06414
	20	0.0299	0.02950	0.03131	0.03342	0.03580
	100	0.0253	0.02531	0.02539	0.02547	0.02557
$\dfrac{h^2\sigma_{12}(0,0,0)}{q_0 a^2}$	4	0.0281	0.02774	0.03233	0.03624	0.03939
	10	0.0120	0.01218	0.01359	0.01514	0.01679
	20	0.0093	0.00929	0.00968	0.01012	0.01062
	100	0.0083	0.00832	0.00834	0.00836	0.00838
$\dfrac{h\sigma_{13}\left(0,\frac{b}{2},\frac{h}{2}\right)}{q_0 a}$	4	0.351	0.32887	0.29089	0.25454	0.22129
	10	0.420	0.41897	0.40979	0.39934	0.38784
	20	0.434	0.43426	0.43181	0.42893	0.42565
	100	0.439	0.43931	0.43921	0.43909	0.43896
$\dfrac{h\sigma_{23}\left(\frac{a}{2},0,\frac{h}{2}\right)}{q_0 a}$	4	0.0334	0.03022	0.03498	0.03901	0.04219
	10	0.0152	0.01483	0.01637	0.01806	0.01988
	20	0.0119	0.01184	0.01227	0.01276	0.01331
	100	0.0108	0.01083	0.01085	0.01087	0.01089

same as given by Di Sciuva (1992). In that paper, as well as the paper by Cho and Parmerter (1993), comparison has been made with an exact three-dimensional elasticity solution and several other plate theories, confirming the high accuracy achieved and the necessity of using the third-order zigzag approach. Therefore, assessment of the present theory for the case of perfect bonding is unnecessary.

To show the overall elastic response of plates, the non-dimensional static central deflection and the first four flexural vibration frequencies are plotted against span-to-thickness ratio in Figs 2–6. It can be seen that due to weakening of the interfacial bond,

Table 2. Frequency $({}^{(1)}\rho h^2/{}^{(2)}E_{1111})^{1/2}\omega$ for the fundamental flexural mode of a square three-layer plate (Material 2)

${}^{(1)}\rho/{}^{(2)}\rho$	${}^{(1)}E_{1111}/{}^{(2)}E_{1111}$	$\bar{R} = 0$ (Exact)	$\bar{R} = 0$	$\bar{R} = 0.2$	$\bar{R} = 0.4$	$\bar{R} = 0.6$
1	1	0.047419	0.047405	0.047340	0.047275	0.047209
1	2	0.057041	0.057024	0.056820	0.056612	0.056398
1	5	0.077148	0.077135	0.076334	0.075511	0.074671
1	10	0.098104	0.098091	0.096136	0.094161	0.092182
1	15	0.112034	0.112021	0.108953	0.105907	0.102909
3	15	0.094548	0.094519	0.091950	0.089397	0.086882

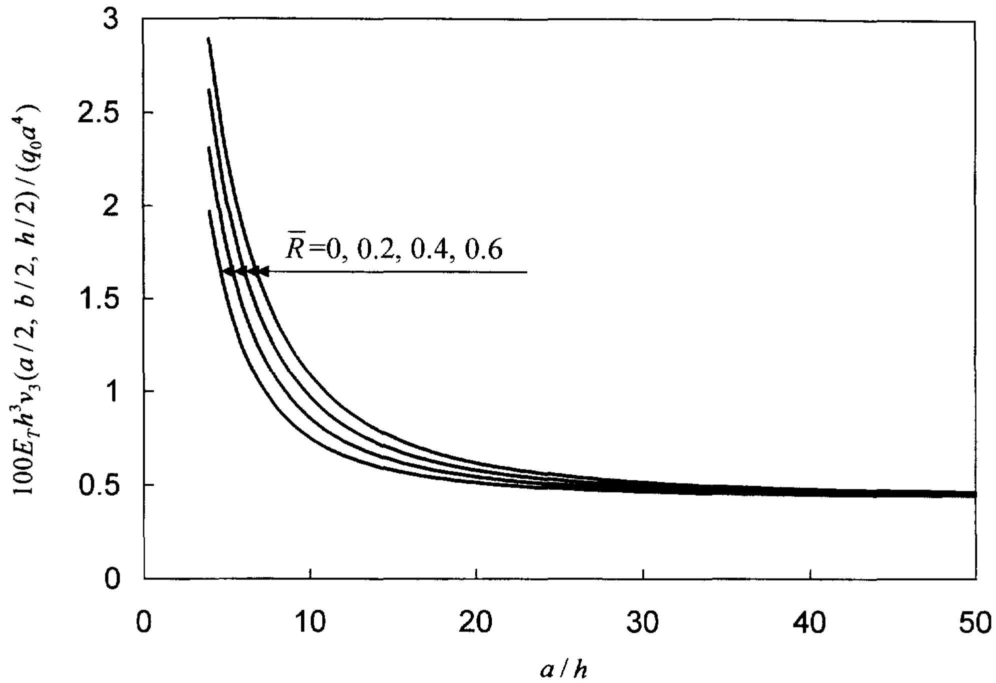

Fig. 2. Effect of span-to-thickness ratio on central deflection.

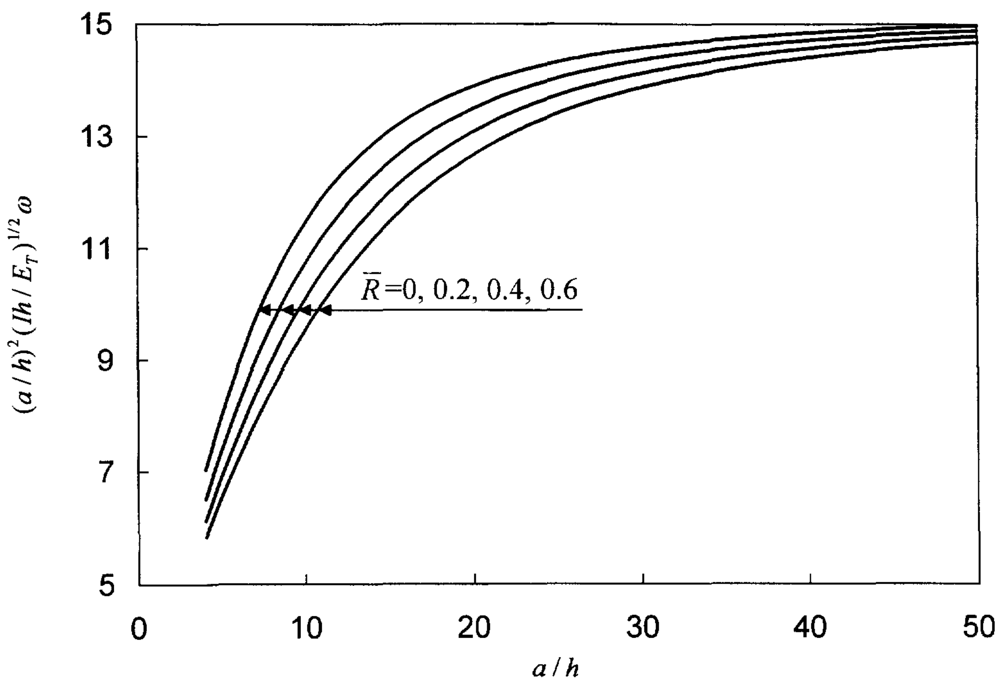

Fig. 3. Effect of span-to-thickness ratio on frequency for the fundamental flexural mode ($m_1 = 1, m_2 = 1$).

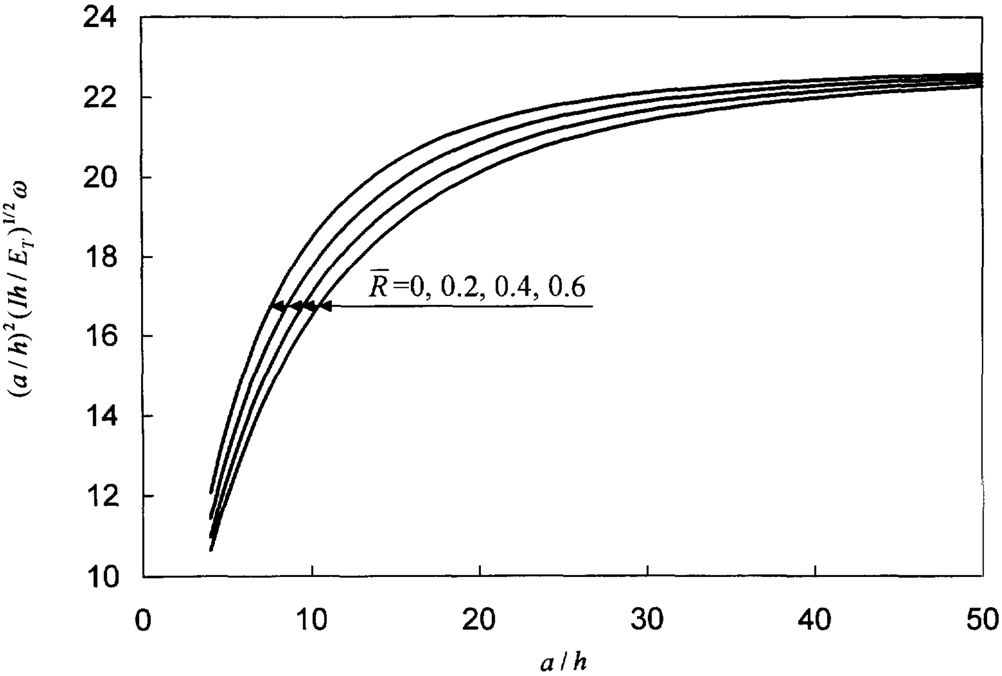

Fig. 4. Effect of span-to-thickness ratio on frequency for the second flexural mode ($m_1 = 1, m_2 = 2$).

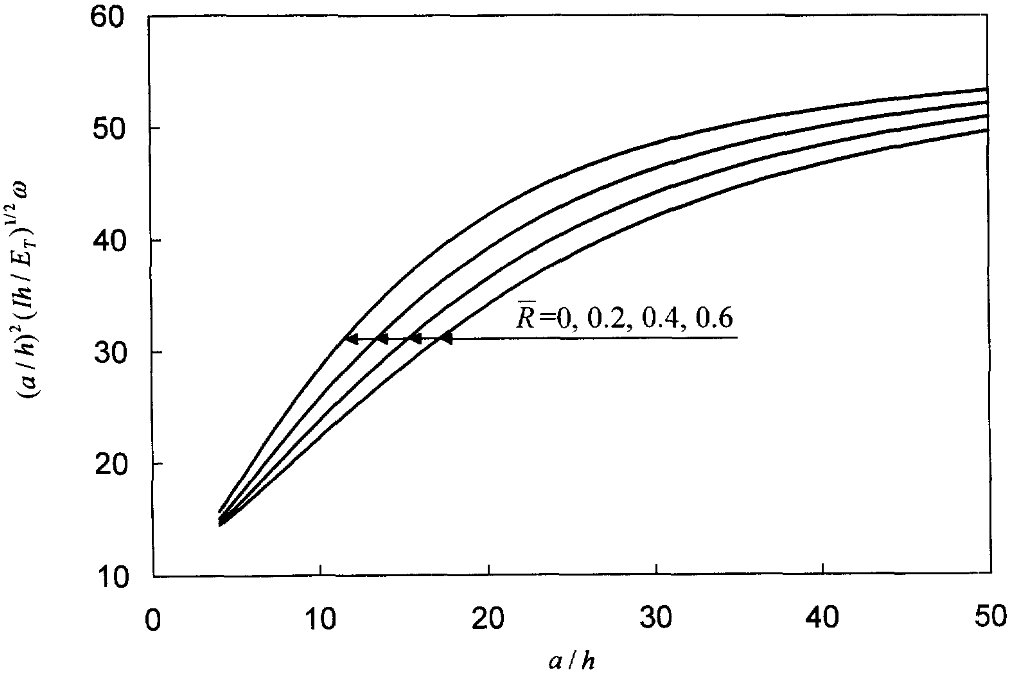

Fig. 5. Effect of span-to-thickness ratio on frequency for the third flexural mode ($m_1 = 2, m_2 = 1$).

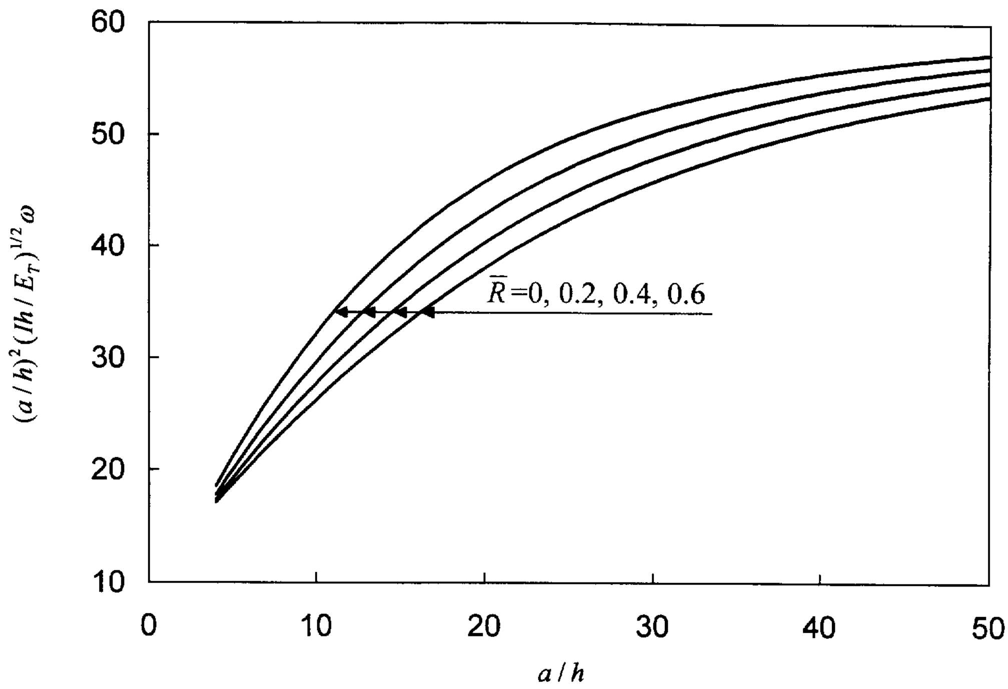

Fig. 6. Effect of span-to-thickness ratio on frequency for the fourth flexural mode ($m_1 = 2, m_2 = 2$).

the rigidity of plates decreases, which leads to increasing central deflection of static bending and decreasing frequencies of flexural vibration, for the same plate configuration. As a/h decreases the static deflection shown in Fig. 2 for bending problems increases faster with larger values of $\bar{R}$, particularly in the range $a/h < 30$. For the vibration problem, it is seen from Figs 3–6 that the first four frequencies decrease with decreasing a/h. To give a better understanding of the way in which local elastic response is affected by weakened bonding, Figs 7–9 show, respectively, the variation of dimensionless inplane displacement, bending stress and transverse shear stress distribution through the plate thickness. In practice, the curing process for certain composites is augmented by introducing a very thin adhesive layer in the interfaces in order to reduce the interlayer stresses, see Mao and Han (1992). Figure 9(a,b) confirms this important phenomenon, i.e. the interlayer stress decreases significantly as the interfacial parameter increases, especially for very thick plates. This is precisely as expected. However, it is also clear from the theoretical prediction that the reduction in interlayer stress is achieved at the expense of increases in overall response.

5. CONCLUSIONS

This paper is devoted to modelling the geometrically nonlinear behaviour of multilayered anisotropic composite plates, with special emphasis on the possibility of incorporating the effects of interfacial imperfection. To do this, each interface between adjacent layers is characterized by a spring-layer model employed in micromechanics. This approach is used in a macro-structural analysis environment to model non-uniform and imperfect bonding at the interfaces of multilayered plates. Uniform bonding is a special case of the theory and results in the governing equations having constant coefficients. The proposed theory has the same advantages as conventional high-order theory. Moreover, it reduces to the zigzag plate theory in the special case of vanishing interface parameters. Numerical examples reveal the important feature that interfacial stresses are reduced by weakening the interfacial bond.

Further work is needed on the estimation of interfacial parameters, either by theoretical prediction of interfacial properties and microstructures or by experiment.

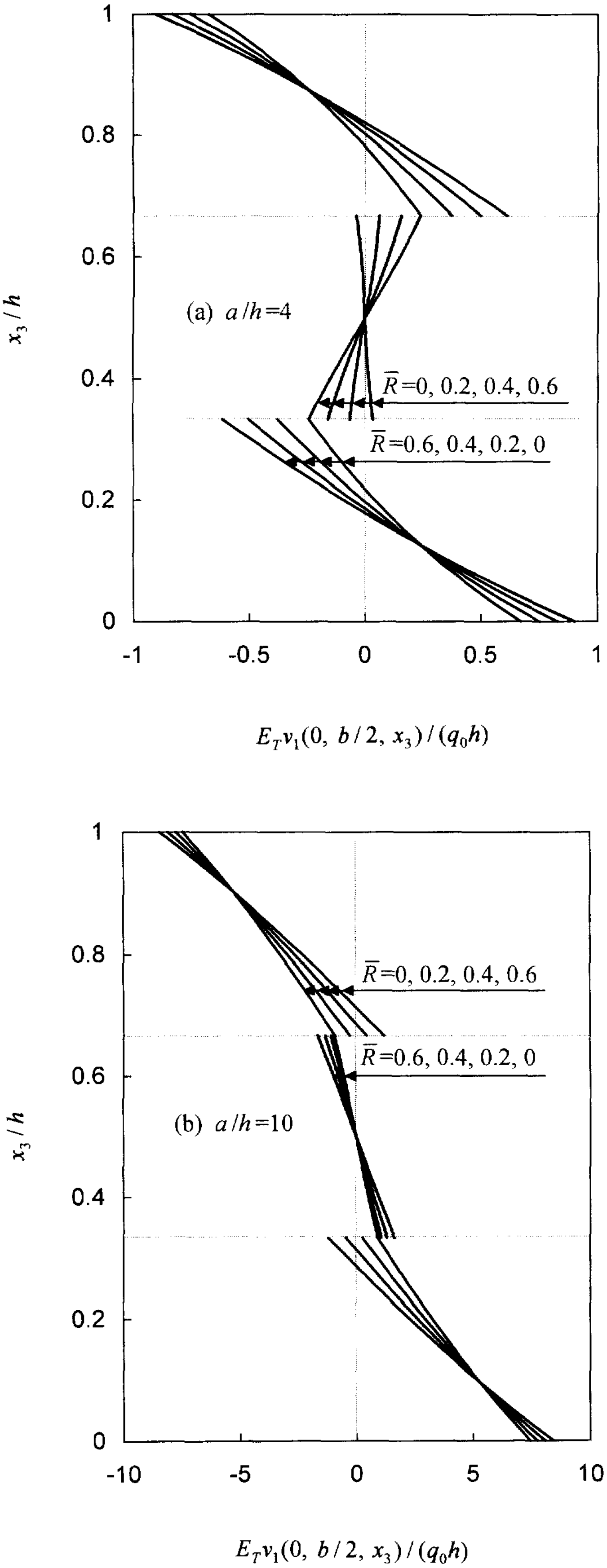

Fig. 7. Variation of inplane displacement through the plate thickness at $(0, b/2, x_3)$.

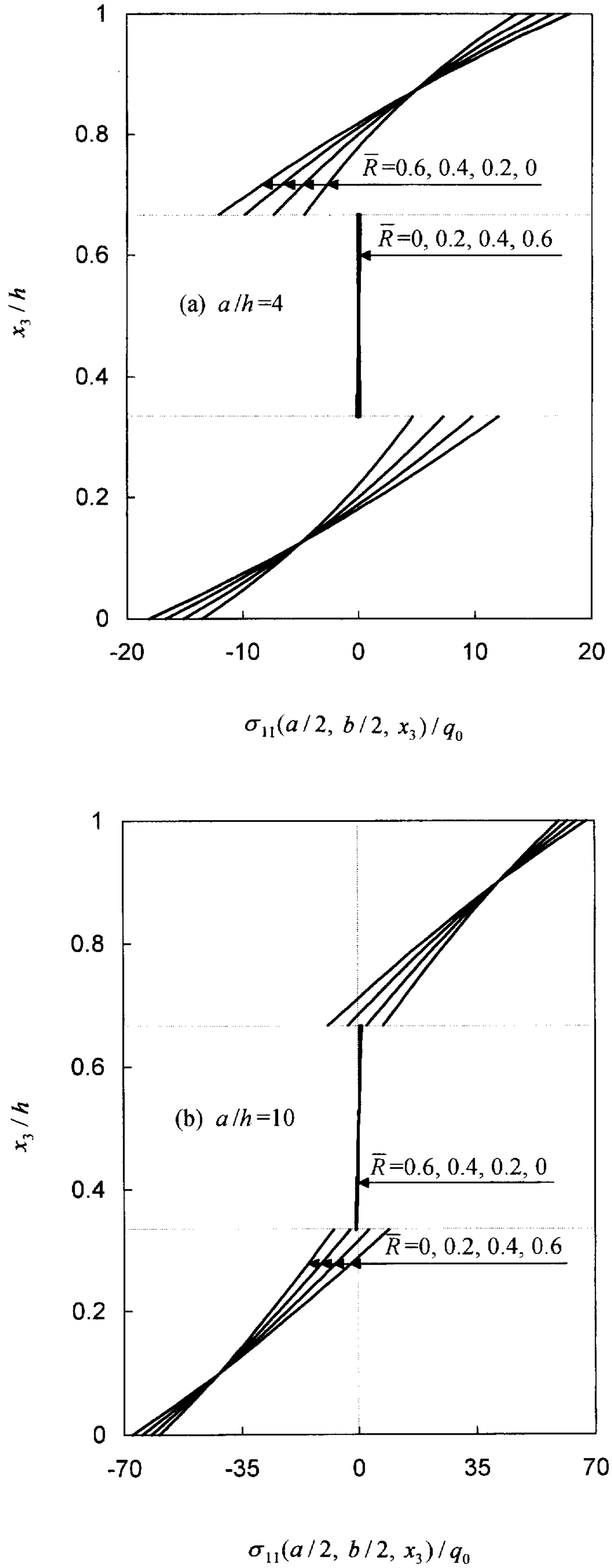

Fig. 8. Bending stress distribution through the plate thickness at $(a/2, b/2, x_3)$.

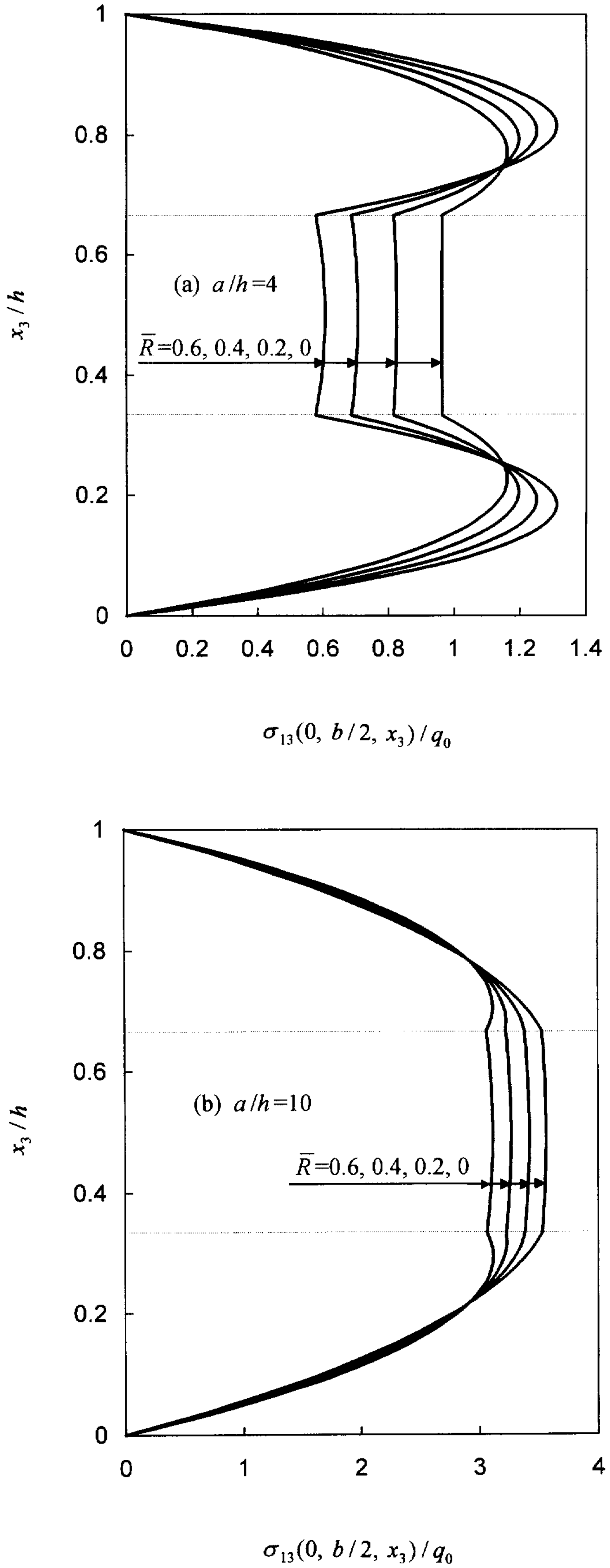

Fig. 9. Transverse shear stress through the plate thickness at $(0, b/2, x_3)$, using equilibrium equations.

Acknowledgements—The first author appreciates the helpful discussions with Professor L. H. He and is grateful to the Cardiff Advanced Chinese Engineering Centre and the National Science Foundation of China for financial support.

REFERENCES

Aboudi, J. (1987). Damage in composites—modelling of imperfect bonding. *Composites Science and Technology* **28,** 103–128.

Achenbach, J. D. and Zhu, H. (1989). Effect of interfacial zone on mechanical behaviour and failure of fibre-reinforced composites. *Journal of Mechanics and Physics of Solids* **37,** 381–393.

Benveniste, Y. (1985). The effective mechanical behaviour of composite materials with imperfect contact between the constituents. *Mechanics of Materials* **4,** 197–208.

Benveniste, Y. and Dvorak, G. J. (1990). On a correspondence between mechanical and thermal fields with slipping interfaces. In *Inelastic Deformation of Composite Materials IUTAM Symposium* (ed. G. J. Dvorak), Springer-Verlag, New York, pp. 77–98.

Chattopadhyay, A. and Gu, H. Z. (1994). New higher order plate theory in modelling delamination buckling of composite laminates. *AIAA Journal* **32,** 1709–1716.

Cheng, Z.-Q., Jemah, A. K. and Williams, F. W. (1996). Theory for multilayered anisotropic plates with weakened interfaces. *Journal of Applied Mechanics* (to appear).

Cho, M. and Parmerter, R. R. (1992). An efficient higher-order plate theory for laminated composites. *Computers & Structures* **20,** 113–123.

Cho, M. and Parmerter, R. R. (1993). Efficient higher order composite plate theory for general lamination configurations. *AIAA Journal* **31,** 1299–1306.

Cho, M. and Parmerter, R. R. (1994). Finite element for composite plate bending based on efficient higher order theory. *AIAA Journal* **32,** 2241–2248

Di Sciuva, M. (1986). Bending, vibration and buckling of simply supported thick multilayered orthotropic plates: an evaluation of a new displacement model. *Journal of Sound and Vibration* **105,** 425–442.

Di Sciuva, M. (1987). An improved shear deformation theory for moderately thick multilayered anisotropic shells and plates. *Journal of Applied Mechanics* **54,** 589–596.

Di Sciuva, M. (1992). Multilayered anisotropic plate models with continuous interlaminar stresses. *Computers & Structures* **22,** 149–167.

Di Sciuva, M. and Icardi, U. (1993). Discrete-layer models for multilayered shells accounting for interlayer continuity. *Meccanica* **28,** 281–291.

Gaudenzi, P. (1992). A general formulation of higher-order theories for the analysis of laminated plates. *Computers & Structures* **20,** 103–112.

Hashin, Z. (1990). Thermoelastic properties of fibre composites with imperfect interface. *Mechanics of Materials* **8,** 333–348.

Hashin, Z. (1991a). Thermoelastic properties of particulate composites with imperfect interface. *Journal of Mechanics and Physics of Solids* **39,** 745–762.

Hashin, Z. (1991b). The spherical inclusion with imperfect interface. *Journal of Applied Mechanics* **58,** 444–449.

Hashin, Z. (1993). Errata on "The spherical inclusion with imperfect interface". *Journal of Applied Mechanics* **60,** 582.

He, L. H. (1993). A novel Karman-type nonlinear theory of anisotropic laminated plates. In *Proceedings of the 2nd International Conference Nonlinear Mechanics* (eds W. Z. Chien), Beijing University Press, Beijing, pp. 237–240.

He, L. H. (1994). A linear theory of laminated shells accounting for continuity of displacements and transverse shear stresses at layer interfaces. *International Journal of Solids and Structures* **31,** 613–627.

He, L. H. (1995). Non-linear theory of laminated shells accounting for continuity conditions of displacements and tractions at layer interfaces. *International Journal of Mechanical Science* **37,** 161–173.

Jasiuk, I. and Tong, Y. (1989). The effect of interface on the elastic stiffness of composites. In *Mechanics of Composite Materials and Structures* (*Proceedings of the Third Joint ASCE/ASME Conference*) (eds J. N. Reddy and J. L. Teply), AMD-Vol. 100, ASME, New York, pp. 49–54.

Librescu, L. (1975). *Elastostatics and Kinetics of Anisotropic and Heterogeneous Shell-type Structures*, Noordhoff, Leyden.

Librescu, L. and Schmidt, R. (1991). Substantiation of a shear-deformable theory of anisotropic composite laminated shells accounting for the interlaminae continuity conditions. *International Journal of Engineering Sciences* **29,** 669–683.

Mao, T. X. and Han, J. H. (1992). Tensile fracture of carbon/epoxy aluminium laminates. In *Proceedings of the Second International Symposium on Computers Materials and Structures* (eds C. T. Sun and T. T. Loo), Beijing University Press, Beijing, pp. 899–904.

Noor, A. K. and Burton, W. S. (1989). Assessment of shear deformation theories for multilayered composite plates. *Applied Mechanics Review* **42,** 1–13.

Pagano, N. J. (1970). Exact solutions for rectangular bidirectional composites and sandwich plates. *Journal of Computers and Materials* **4,** 20–34.

Qu, J. M. (1993a). Eshelby tensor for an elastic inclusion with slightly weakened interfaces. *Journal of Applied Mechanics* **60,** 1048–1050.

Qu, J. M. (1993b). The effect of slightly weakened interfaces on the overall elastic properties of composite materials. *Mechanics of Materials* **14,** 269–281.

Reddy, J. N. (1984). A simple higher-order theory for laminated composite plates. *Journal of Applied Mechanics* **51,** 745–752.

Reddy, J. N. and Robbins Jr, D. H. (1994). Theories and computational models for composite laminates. *Applied Mechanics Review* **47,** 147–169.

Savithri, S. and Varadan, T. K. (1990). Accurate bending analysis of laminated orthotropic plates. *AIAA Journal* **28,** 1842–1844.
Savithri, S. and Varadan, T. K. (1993). Large deflection analysis of laminated composite plates. *International Journal of Non-Linear Mechanics* **28,** 1–12.
Schmidt, R. and Librescu, L. (1994). Further results concerning the refined theory of anisotropic laminated composite plates. *Journal of Engineering Mathematics* **28,** 407–425.
Schmidt, R. and Librescu, L. (1995). A refined theory of anisotropic laminated composite plates featuring nonrigidly bonded interfaces. Presented at *15th Canadian Congress on Applied Mechanics*.
Srinivas, S. and Rao, A. K. (1970). Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. *International Journal of Solids and Structures* **6,** 1483–1481.
Xavier, P. B., Lee, K. H. and Chew, C. H. (1993). An improved zigzag model for the bending of laminated composite shells. *Computers & Structures* **26,** 123–138.

APPENDIX A

In order to develop a practical theory of multilayered plates, which can model slightly weakened interfaces but not debonding, eqn (1) is truncated by using the approximations

$$n = \begin{cases} 0,1,2,3 & \text{for } j=\alpha \text{ and } m=0 \\ 0,1 & \text{for } j=\alpha \text{ and } m=1,\ldots,k-1, \\ 0 & \text{for } j=3 \text{ and } m=0,\ldots,k-1 \end{cases} \tag{A1}$$

the displacements can be expressed as

$$v_\alpha(x_i;t) = u_\alpha + \psi_\alpha x_3 + \varphi_\alpha x_3^2 + \eta_\alpha x_3^3 + \sum_{m=1}^{k-1} [{}^{(m)}\Delta v_\alpha + {}^{(m)}u_\alpha(x_3 - {}^{(m)}h)]H(x_3 - {}^{(m)}h),$$

$$v_3(x_i;t) = u_3 + \sum_{m=1}^{k-1} {}^{(m)}\Delta v_3 H(x_3 - {}^{(m)}h), \tag{A2a,b}$$

where ${}^{(0)}u_j^{(0)}$, ${}^{(0)}u_\alpha^{(1)}$, ${}^{(0)}u_\alpha^{(2)}$, ${}^{(0)}u_\alpha^{(3)}$, ${}^{(m)}u_j^{(0)}$ and ${}^{(m)}u_\alpha^{(1)}$ in eqn (1) have been replaced by the quantities u_j, ψ_α, φ_α, η_α, ${}^{(m)}\Delta v_j$ and ${}^{(m)}u_\alpha$, respectively. Of course, theories developed for calculating delamination need more terms than are retained by eqns (A2), e.g., see Chattopadhyay and Gu (1994).

The strain, in the sense of Karman large deflection, and stress components of the plate can be obtained from

$$e_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i} + v_{3,i}v_{3,j}), \quad \sigma_{\alpha\beta} = H_{\alpha\beta\omega\rho}e_{\omega\rho}, \quad \sigma_{\alpha3} = 2E_{\alpha3\omega3}e_{\omega3}, \tag{A3a–c}$$

where e_{ij} and σ_{ij} are components of the strain and stress tensors, E_{ijkl} are components of the elasticity tensor associated with an elastic anisotropic body, and

$$H_{\alpha\beta\omega\rho} = E_{\alpha\beta\omega\rho} - \frac{E_{\alpha\beta33}E_{33\omega\rho}}{E_{3333}}. \tag{A4}$$

Here, as indicated by Librescu (1975), eqns (A3b,c) hold valid only under the assumptions that each layer possesses a plane of elastic symmetry parallel to the $x_3 = 0$ plane and that σ_{33} is vanishingly small compared with the other components of the stress tensor.

The assumption of vanishing σ_{33} leads to, from eqn (4b),

$${}^{(m)}\Delta v_3 = 0, \quad (m = 1,\ldots,k-1), \tag{A5}$$

which then gives, from eqn (A2b),

$$v_3(x_i;t) = u_3. \tag{A6}$$

The compatibility conditions of transverse shear stresses on the two bounding surfaces of the plate as well as at the interfaces are now used to reduce the number of unknowns in eqns (A2a). For simplicity, it is assumed that no tangential tractions are exerted on ${}^{(0)}\Omega$ and ${}^{(k)}\Omega$, where eqns (A2a), (A6) and (A3a,c) give the tangential tractions. Hence

$$\psi_\alpha = -u_{3,\alpha}, \quad \eta_\alpha = -\frac{2}{3}\left(\frac{1}{h}\varphi_\alpha + \frac{1}{2h^2}\sum_{m=1}^{k-1} {}^{(m)}u_\alpha\right). \tag{A7}$$

The condition (3a) for continuously distributed transverse shear stresses at the interfaces leads to, by using eqns (A2a), (A6), (A7) and (A3a,c),

$$\frac{1}{2}{}^{(i)}E_{\alpha3\omega3}{}^{(i)}u_\omega + ({}^{(i+1)}E_{\alpha3\omega3} - {}^{(i)}E_{\alpha3\omega3})\left[\left({}^{(i)}h - \frac{1}{h}{}^{(i)}h^2\right)\varphi_\omega + \frac{1}{2}\sum_{m=1}^{i} {}^{(m)}u_\omega - \frac{1}{2h^2}{}^{(i)}h^2\sum_{m=1}^{k-1} {}^{(m)}u_\omega\right] = 0,$$

$$(i = 1,\ldots,k-1). \tag{A8}$$

In fact, eqn (A8) can be regarded as $2(k-1)$ linear algebraic equations involving the $2(k-1)$ unknowns ${}^{(i)}u_\alpha$ ($i = 1, \ldots, k-1$), which give the following relationship between ${}^{(i)}u_\alpha$ and φ_λ

$$ {}^{(i)}u_\alpha = {}^{(i)}a_{\alpha\lambda}\varphi_\lambda, \quad (i = 1, \ldots, k-1), \tag{A9} $$

in which the ${}^{(i)}a_{\alpha\lambda}$ depend only on the material elasticity properties of each layer and are therefore known constants.

Substitution of eqns (A7) and (A9) into eqn (A2a) yields

$$ v_\alpha = u_\alpha - x_3 u_{3,\alpha} + f_{\alpha\lambda}\varphi_\lambda + \sum_{m=1}^{k-1} {}^{(m)}\Delta v_\alpha H(x_3 - {}^{(m)}h), \tag{A10} $$

in which, using the Kronecker delta,

$$ f_{\alpha\lambda} \equiv f_{\alpha\lambda}(x_3) = \delta_{\alpha\lambda}x_3^2 - \frac{2}{3h}\left(\delta_{\alpha\lambda} + \frac{1}{2h}\sum_{m=1}^{k-1} {}^{(m)}a_{\alpha\lambda}\right)x_3^3 + \sum_{m=1}^{k-1} {}^{(m)}a_{\alpha\lambda}(x_3 - {}^{(m)}h)H(x_3 - {}^{(m)}h). \tag{A11} $$

APPENDIX B

Closed form solutions to the example of Section 4 can be found by substituting eqns (20) into the linear counterpart of eqns (19), which yields

$$ \mathbf{AX} = \mathbf{F}, \tag{B1} $$

where $\mathbf{X} = [U_1 \; U_2 \; U_3 \; \Phi_1 \; \Phi_2]^T$, $\mathbf{F} = [0 \; 0 \; -q_0 \; 0 \; 0]^T$, and $\mathbf{A}$ is a 5×5 symmetric matrix ($A_{IJ} = A_{JI}$, $I, J = 1, \ldots, 5$) where its elements, expressed in terms of $l_1 = m_1\pi/a$ and $l_2 = m_2\pi/b$, are

$$
\begin{aligned}
A_{11} &= -l_1^2C^{(1)}_{1111} - l_2^2C^{(1)}_{1212} + I\omega^2,\\
A_{12} &= -l_1l_2C^{(1)}_{1122} - l_1l_2C^{(1)}_{1221},\\
A_{13} &= l_1^3C^{(2)}_{1111} + l_1l_2^2C^{(2)}_{1122} + l_1l_2^2(C^{(2)}_{1212} + C^{(2)}_{1221}) - l_1J\omega^2,\\
A_{14} &= -l_1^2C^{(3)}_{1111} - l_2^2C^{(3)}_{1212} + I_{11}\omega^2,\\
A_{15} &= -l_1l_2C^{(3)}_{1122} - l_1l_2C^{(3)}_{1221},\\
A_{22} &= -l_2^2C^{(1)}_{2222} - l_1^2C^{(1)}_{2121} + I\omega^2,\\
A_{23} &= l_1^2l_2C^{(2)}_{2211} + l_2^3C^{(2)}_{2222} + l_1^2l_2(C^{(2)}_{2112} + C^{(2)}_{2121}) - l_2J\omega^2,\\
A_{24} &= -l_1l_2C^{(3)}_{2211} - l_1l_2C^{(3)}_{2112},\\
A_{25} &= -l_2^2C^{(3)}_{2222} - l_1^2C^{(3)}_{2121} + I_{22}\omega^2,\\
A_{33} &= -l_1^4C^{(4)}_{1111} - l_1^2l_2^2(C^{(4)}_{1122} + C^{(4)}_{2211}) - l_2^4C^{(4)}_{2222} - l_1^2l_2^2(C^{(4)}_{1212} + C^{(4)}_{1221} + C^{(4)}_{2112} + C^{(4)}_{2121})\\
&\quad + I\omega^2 + (l_1^2 + l_2^2)K\omega^2,\\
A_{34} &= l_1^3C^{(5)}_{1111} + l_1l_2^2C^{(5)}_{2211} + l_1l_2^2(C^{(5)}_{1212} + C^{(5)}_{2112}) - l_1J_{11}\omega^2,\\
A_{35} &= l_1^2l_2C^{(5)}_{1122} + l_2^3C^{(5)}_{2222} + l_1^2l_2(C^{(5)}_{1221} + C^{(5)}_{2121}) - l_2J_{22}\omega^2,\\
A_{44} &= -l_1^2C^{(6)}_{1111} - l_2^2C^{(6)}_{1212} - C^{(8)}_{11} + K_{11}\omega^2,\\
A_{45} &= -l_1l_2C^{(6)}_{1122} - l_1l_2C^{(6)}_{1221},\\
A_{55} &= -l_2^2C^{(6)}_{2222} - l_1^2C^{(6)}_{2121} - C^{(8)}_{22} + K_{22}\omega^2.
\end{aligned}
\tag{B2}
$$